What does a matrix look like?

Matrices are everywhere. If you have used a spreadsheet such as Excel or written numbers in a table, you have used a matrix. Matrices make presentation of numbers clearer and make calculations easier to program. Look at the matrix below about the sale of tires in a Blowoutr'us store – given by quarter and make of tires.

$$
\begin{array}{cccc}
& Q1 & Q2 & Q3 & Q4 \\
\text{Tirestone} & \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \\
\text{Michigan} & \\
\text{Copper} &
\end{array}
$$

If one wants to know how many *Copper* tires were sold in *Quarter* 4, we go along the row *Copper* and column *Q4* and find that it is 27.

So what is a matrix?

A *matrix* is a rectangular array of elements. The elements can be symbolic expressions or/and numbers. Matrix $[A]$ is denoted by

$$
[A] = \begin{bmatrix}
a_{11} & a_{12} & \cdots\cdots & a_{1n} \\
a_{21} & a_{22} & \cdots\cdots & a_{2n} \\
\vdots & & & \vdots \\
a_{m1} & a_{m2} & \cdots\cdots & a_{mn}
\end{bmatrix}
$$

Row i of $[A]$ has n elements and is

$$
\begin{bmatrix} a_{i1} & a_{i2}....a_{in} \end{bmatrix}
$$

and column j of $[A]$ has m elements and is

$$
\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}
$$

Each matrix has rows and columns and this defines the size of the matrix. If a matrix $[A]$ has m rows and n columns, the size of the matrix is denoted by $m \times n$. The matrix $[A]$ may also be denoted by $[A]_{m \times n}$ to show that $[A]$ is a matrix with m rows and n columns.

Each entry in the matrix is called the entry or element of the matrix and is denoted by a_{ij} where i is the row number and j is the column number of the element.

The matrix for the tire sales example could be denoted by the matrix $[A]$ as

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}.$$

There are 3 rows and 4 columns, so the size of the matrix is 3×4. In the above $[A]$ matrix, $a_{34} = 27$.

What are the special types of matrices?

Vector: A vector is a matrix that has only one row or one column. There are two types of vectors – row vectors and column vectors.

Row Vector:

If a matrix $[B]$ has one row, it is called a row vector $[B] = [b_1 \ b_2 \ \ldots \ldots b_n]$ and n is the dimension of the row vector.

Example 1

Give an example of a row vector.
Solution

$$[B] = [25 \ 20 \ 3 \ 2 \ 0]$$

is an example of a row vector of dimension 5.

Column vector:

If a matrix $[C]$ has one column, it is called a column vector

$$[C] = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}$$

and m is the dimension of the vector.

Example 2

Give an example of a column vector.
Solution

$$[C] = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix}$$

is an example of a column vector of dimension 3.

Submatrix:

If some row(s) or/and column(s) of a matrix $[A]$ are deleted (no rows or columns may be deleted), the remaining matrix is called a submatrix of $[A]$.

Example 3

Find some of the submatrices of the matrix

$$[A] = \begin{bmatrix} 4 & 6 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 4 & 6 & 2 \\ 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 6 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 6 & 2 \end{bmatrix}, \begin{bmatrix} 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

are some of the submatrices of $[A]$. Can you find other submatrices of $[A]$?

Square matrix:

If the number of rows m of a matrix is equal to the number of columns n of a matrix $[A]$, that is, $m = n$, then $[A]$ is called a square matrix. The entries $a_{11}, a_{22},..., a_{nn}$ are called the *diagonal elements* of a square matrix. Sometimes the diagonal of the matrix is also called the *principal* or *main of the matrix.*

Example 4

Give an example of a square matrix.
Solution

$$[A] = \begin{bmatrix} 25 & 20 & 3 \\ 5 & 10 & 15 \\ 6 & 15 & 7 \end{bmatrix}$$

is a square matrix as it has the same number of rows and columns, that is, 3. The diagonal elements of $[A]$ are $a_{11} = 25$, $a_{22} = 10$, $a_{33} = 7$.

Upper triangular matrix:

A $n \times n$ matrix for which $a_{ij} = 0$, $i > j$ for all i, j is called an upper triangular matrix. That is, all the elements below the diagonal entries are zero.

Example 5

Give an example of an upper triangular matrix.
Solution

$$[A] = \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix}$$

is an upper triangular matrix.

Lower triangular matrix:

A $n \times n$ matrix for which $a_{ij} = 0,\ j > i$ for all i, j is called a lower triangular matrix. That is, all the elements above the diagonal entries are zero.

Example 6

Give an example of a lower triangular matrix.
Solution

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0.6 & 2.5 & 1 \end{bmatrix}$$

is a lower triangular matrix.

Diagonal matrix:

A square matrix with all non-diagonal elements equal to zero is called a diagonal matrix, that is, only the diagonal entries of the square matrix can be non-zero, ($a_{ij} = 0, i \neq j$).

Example 7

Give examples of a diagonal matrix.
Solution

$$[A] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is a diagonal matrix.
Any or all the diagonal entries of a diagonal matrix can be zero. For example

$$[A] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is also a diagonal matrix.

Identity matrix:

A diagonal matrix with all diagonal elements equal to 1 is called an identity matrix, ($a_{ij} = 0, i \neq j$ for all i, j and $a_{ii} = 1$ for all i).

Example 8

Give an example of an identity matrix.
Solution

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix.

Zero matrix:

A matrix whose all entries are zero is called a zero matrix, ($a_{ij} = 0$ for all i and j).

Example 9

Give examples of a zero matrix.
Solution

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

are all examples of a zero matrix.

Tridiagonal matrices:

A tridiagonal matrix is a square matrix in which all elements not on the following are zero - the major diagonal, the diagonal above the major diagonal, and the diagonal below the major diagonal.

Example 10

Give an example of a tridiagonal matrix.
Solution

$$[A] = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 2 & 3 & 9 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

is a tridiagonal matrix.

Do non-square matrices have diagonal entries?

Yes, for a $m \times n$ matrix $[A]$, the diagonal entries are $a_{11}, a_{22}..., a_{k-1,k-1}, a_{kk}$ where $k = \min\{m, n\}$.

Example 11

What are the diagonal entries of

$$[A] = \begin{bmatrix} 3.2 & 5 \\ 6 & 7 \\ 2.9 & 3.2 \\ 5.6 & 7.8 \end{bmatrix}$$

Solution

The diagonal elements of $[A]$ are $a_{11} = 3.2$ and $a_{22} = 7$.

Diagonally Dominant Matrix:

A $n \times n$ square matrix $[A]$ is a diagonally dominant matrix if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \text{ for } i = 1,2,.....,n \text{ and}$$

$$|a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \text{ for at least one } i,$$

that is, for each row, the absolute value of the diagonal element is greater than or equal to the sum of the absolute values of the rest of the elements of that row, and that the inequality is strictly

greater than for at least one row. Diagonally dominant matrices are important in ensuring convergence in iterative schemes of solving simultaneous linear equations.

Example 12

Give examples of diagonally dominant matrices and not diagonally dominant matrices.
Solution

$$[A] = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4 & -2 \\ 3 & 2 & 6 \end{bmatrix}$$

is a diagonally dominant matrix as

$$|a_{11}| = |15| = 15 \geq |a_{12}| + |a_{13}| = |6| + |7| = 13$$
$$|a_{22}| = |-4| = 4 \geq |a_{21}| + |a_{23}| = |2| + |-2| = 4$$
$$|a_{33}| = |6| = 6 \geq |a_{31}| + |a_{32}| = |3| + |2| = 5$$

and for at least one row, that is Rows 1 and 3 in this case, the inequality is a strictly greater than inequality.

$$[B] = \begin{bmatrix} -15 & 6 & 9 \\ 2 & -4 & 2 \\ 3 & -2 & 5.001 \end{bmatrix}$$

is a diagonally dominant matrix as

$$|b_{11}| = |-15| = 15 \geq |b_{12}| + |b_{13}| = |6| + |9| = 15$$
$$|b_{22}| = |-4| = 4 \geq |b_{21}| + |b_{23}| = |2| + |2| = 4$$
$$|b_{33}| = |5.001| = 5.001 \geq |b_{31}| + |b_{32}| = |3| + |-2| = 5$$

The inequalities are satisfied for all rows and it is satisfied strictly greater than for at least one row (in this case it is Row 3).

$$[C] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is not diagonally dominant as
$$|c_{22}| = |8| = 8 \leq |c_{21}| + |c_{23}| = |64| + |1| = 65$$

When are two matrices considered to be equal?

Two matrices $[A]$ and $[B]$ are equal if the size of $[A]$ and $[B]$ is the same (number of rows and columns of $[A]$ are same as that of $[B]$) and $a_{ij} = b_{ij}$ for all i and j.

Example 13

What would make
$$[A] = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$$
to be equal to
$$[B] = \begin{bmatrix} b_{11} & 3 \\ 6 & b_{22} \end{bmatrix}$$

Solution

The two matrices $[A]$ and $[B]$ ould be equal if $b_{11} = 2$ and $b_{22} = 7$.

ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS OF EQATIONS:

Iterative Solution Methods

- Starts with an initial approximation for the solution vector (x^0)
- At each iteration updates the x vector by using the sytem Ax=b
- During the iterations A, matrix is not changed so sparcity is preserved
- Each iteration involves a matrix-vector product
- If A is sparse this product is efficiently done

Iterative solution procedure

- Write the system Ax=b in an equivalent form

  x=Ex+f (like x=g(x) for fixed-point iteration)

- Starting with x^0, generate a sequence of approximations $\{x^k\}$ iteratively by

  $x^{k+1}=Ex^k+f$

- Representation of E and f depends on the type of the method used

- But for every method E and f are obtained from A and b, but in a different way

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Convergence

- As $k\rightarrow\infty$, the sequence $\{x^k\}$ converges to the solution vector under some conditions on E matrix

- This imposes different conditions on A matrix for different methods

- For the same A matrix, one method may converge while the other may diverge

- Therefore for each method the relation between A and E should be found to decide on the convergence

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Iterative Methods: For Example
1) Jacobi Method
2) Gauss Seidel Method

Jacobi Method

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

$$x_1^1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2^0 - \cdots - a_{1n}x_n^0)$$

$$x_2^1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1^0 - a_{23}x_3^0 - \cdots - a_{2n}x_n^0)$$

$$x_n^1 = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^0 - a_{n2}x_2^0 - \cdots - a_{nn-1}x_{n-1}^0)$$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^k \right]$$

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$x^{k+1} = Ex^k + f$ iteration for Jacobi method

A can be written as A=L+D+U (*not decomposition*)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$$Ax=b \implies (L+D+U)x=b \qquad Dx^{k+1} = -(L+U)x^k + b$$

$$x_i^{k+1} = \underbrace{\frac{1}{a_{ii}}}_{Dx^{k+1}}\left[b_i - \underbrace{\sum_{j=1}^{i-1} a_{ij}x_j^k}_{Lx^k} - \underbrace{\sum_{j=i+1}^{n} a_{ij}x_j^k}_{Ux^k} \right]$$

$$x^{k+1} = -D^{-1}(L+U)x^k + D^{-1}b$$

$$E = -D^{-1}(L+U)$$

$$f = D^{-1}b$$

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ALGORITHM OF JACOBI METHOD:

Jacobi Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT   the number of equations and unknowns n; the entries a_{ij}, $1 \leq i, j \leq n$ of the matrix A; the entries b_i, $1 \leq i \leq n$ of $\mathbf{b}$; the entries XO_i, $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT   the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1   Set $k = 1$.

Step 2   While ($k \leq N$) do Steps 3–6.

   Step 3   For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{-\sum\limits_{\substack{j=1 \\ j \neq i}}^{n}(a_{ij}XO_j) + b_i}{a_{ii}}.$$
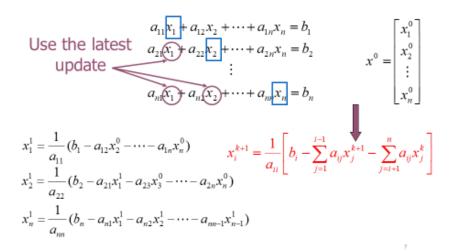
   Step 4   If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$;
                     (*Procedure completed successfully.*)
                     STOP.

   Step 5   Set $k = k + 1$.

   Step 6   For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7   OUTPUT ('Maximum number of iterations exceeded');
         (*Procedure completed unsuccessfully.*)
          STOP.

Gauss-Seidel Method

Use the latest update →

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

$$x_1^1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2^0 - \cdots - a_{1n}x_n^0)$$

$$x_2^1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1^1 - a_{23}x_3^0 - \cdots - a_{2n}x_n^0)$$

$$x_n^1 = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^1 - a_{n2}x_2^1 - \cdots - a_{nn-1}x_{n-1}^1)$$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k \right]$$

$x^{(k+1)} = Ex^{(k)} + f$ iteration for Gauss-Seidel

$$Ax = b \Rightarrow (L+D+U)x = b$$

$$\underbrace{x_i^{k+1}}_{} = \frac{1}{a_{ii}}\left[b_i - \underbrace{\sum_{j=1}^{i-1} a_{ij}x_j^{k+1}}_{} - \underbrace{\sum_{j=i+1}^{n} a_{ij}x_j^k}_{} \right]$$

Dx^{k+1} Lx^{k+1} Ux^k

$$(D+L)x^{k+1} = -Ux^k + b$$

$$x^{k+1} = -(D+L)^{-1}Ux^k + (D+L)^{-1}b$$

$$E = -(D+L)^{-1}U$$

$$f = -(D+L)^{-1}b$$

ALGORITHM OF GAUSS SEIDEL METHOD :

Gauss-Seidel Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT   the number of equations and unknowns n; the entries a_{ij}, $1 \le i, j \le n$ of the matrix A; the entries b_i, $1 \le i \le n$ of $\mathbf{b}$; the entries XO_i, $1 \le i \le n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT   the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1   Set $k = 1$.

Step 2   While ($k \le N$) do Steps 3–6.

   Step 3   For $i = 1, \ldots, n$

   $$\text{set } x_i = \frac{-\sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}XO_j + b_i}{a_{ii}}.$$

   Step 4   If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$;
                           (*Procedure completed successfully.*)
                           STOP.

   Step 5   Set $k = k + 1$.

   Step 6   For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7   OUTPUT ('Maximum number of iterations exceeded');
           (*Procedure completed unsuccessfully.*)
           STOP.

Comparison

- Gauss-Seidel iteration converges more rapidly than the Jacobi iteration since it uses the latest updates

- But there are some cases that *Jacobi* iteration *does converge* but *Gauss-Seidel* does *not*